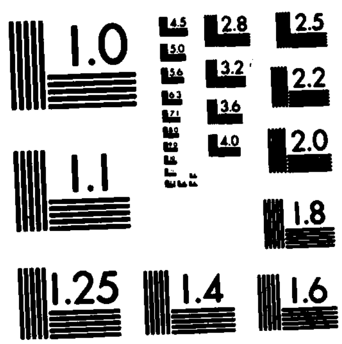

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Bethesda, Maryland 20084

COMPUTER IMPLEMENTATION OF COUPLED BOUNDARY INTEGRAL EQUATION AND FINITE ELEMENT METHODS

by

Erwin A. Schroeder

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COMPUTATION, MATHEMATICS AND LOGISTICS DEPARTMENT
RESEARCH AND DEVELOPMENT REPORT

February 1983

DTNSRDC-83/002

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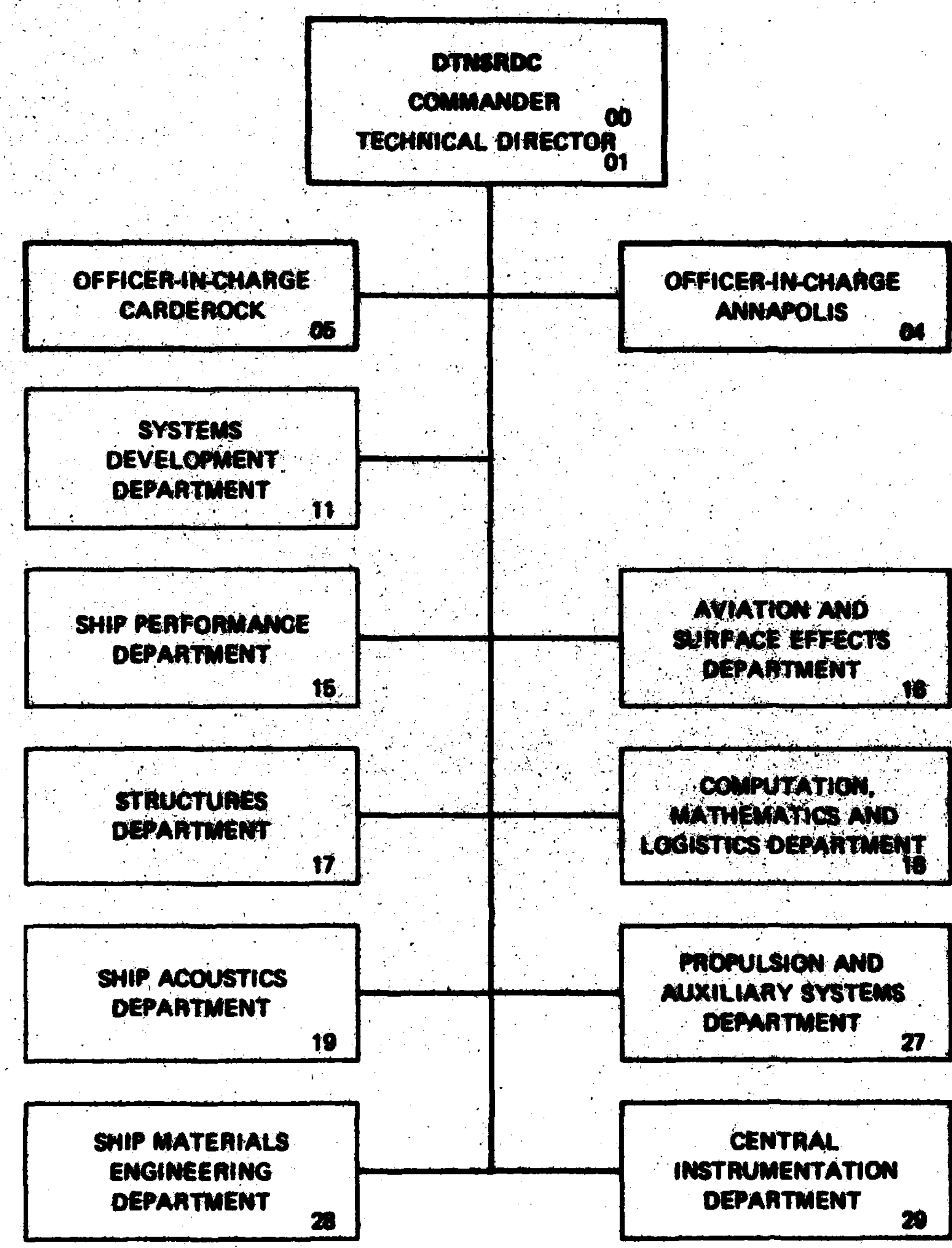

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER DTNSRDC-83/002	2. GOVT ACCESSION NO. AD-A125 074	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) COMPUTER IMPLEMENTATION OF COUPLED BOUNDARY INTEGRAL EQUATION AND FINITE ELEMENT METHODS		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Erwin A. Schroeder		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS David Taylor Naval Ship Research and Development Center Bethesda, Maryland 20084		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (See reverse side)
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE February 1983 13. NUMBER OF PAGES 33
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Sea Systems Command (S03R22) Washington, D.C. 20360		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Boundary Integral Equation
Infinite Regions
Coupled Methods
Finite Element

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes a method for computing solutions to partial differential equations in unbounded regions. The differential equations and unbounded regions arise in problems such as underwater acoustic and magnetic or electric fields surrounding vehicles or instruments in the sea or air. The method couples finite element and boundary integral equation solutions to compute shock response of submarine hulls, acoustic scattering

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Task Area SR0140301
Task 15321
Work Unit 1808-010

(Block 20 continued)

due to elastic structures, electromagnetic signatures of ships and submarines, etc. The finite element method is used in the bounded region occupied by the vehicle or instrument. In this region there may be acoustic, magnetic or electric sources and the material properties may be nonuniform. The boundary integral equation method is used in the unbounded region occupied by the surrounding sea or atmosphere. This method requires that the unbounded region be assumed to be free of sources and have uniform material properties. The coupled method produced good results for a sample problem in which the temperature distribution was computed for a two-dimensional steady state heat flow.

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ABSTRACT

This report describes a method for computing solutions to partial differential equations in unbounded regions. The differential equations and unbounded regions arise in problems such as underwater acoustic and magnetic or electric fields surrounding vehicles or instruments in the sea or air. The method couples finite element and boundary integral equation solutions to compute shock response of submarine hulls, acoustic scattering due to elastic structures, electromagnetic signatures of ships and submarines, etc. The finite element method is used in the bounded region occupied by the vehicle or instrument. In this region there may be acoustic, magnetic or electric sources and the material properties may be nonuniform. The boundary integral equation method is used in the unbounded region occupied by the surrounding sea or atmosphere. This method requires that the unbounded region be assumed to be free of sources and have uniform material properties. The coupled method produced good results for a sample problem in which the temperature distribution was computed for a two-dimensional steady state heat flow.

ADMINISTRATIVE INFORMATION

This work was funded by the Naval Sea Systems Command (SO3R22) under Task Area SR0140301, Task 15321 and Work Unit 1808-010.

I. INTRODUCTION

Several problems of interest to the Navy involve a vehicle surrounded by a medium of large extent. Two examples are:

- Sound radiation or scattering from a submerged structure. The structure may have a complicated shape, including appendages, and the surrounding water is assumed to have uniform acoustical properties.
- Electric or magnetic fields surrounding a ship. The ship may have an irregular shape and may have machinery and materials with differing magnetic properties and machinery that influences the fields, but the surrounding air or water is assumed to have uniform electric and magnetic properties.

The characteristic feature of such problems is that the vehicle is relatively small and may have a complex configuration, while the surrounding medium has a large extent and is assumed to have uniform properties. These problems present two contrasting requirements. There is a small inner region that requires a detailed representation and an outer region that does not need detailed representation but has a large extent. The outer region is often so large that its extent is assumed to be infinite.

This report presents a method for satisfying these two requirements by coupling two solution methods. In the finite element method used to provide a detailed analysis of the inner region, the inner region is divided into a number of elements so that local changes in material properties and local sources can be represented in the elements. The boundary integral equation method, used to provide the analysis of the outer infinite region, reduces the problem of solving a field problem in the external region to a much smaller problem on just the interface between the inner and outer regions. The problem is discretized by dividing the interface into a number of segments. A requirement of using the boundary integral equation method in the form described in this report is that the outer region can be assumed to be free of sources (acoustic, magnetic, etc.) and to have uniform material properties.

This report avoids the complications of a general development by addressing a specific problem, namely steady state heat flow in two dimensions. Two formulations of the coupling are given. In one formulation an unsymmetric system of equations derived by the boundary integral equation method is coupled to the symmetric finite element system of equations, producing an unsymmetric system of equations to be solved. In the other formulation, with additional effort, a symmetric system is derived by the boundary integral equation method, which, when coupled to the finite element equations, produces a symmetric system of equations to be solved.

A sample problem was solved using each method. In this sample problem the temperature distribution was computed for steady state heat conduction in two-dimensions. Both methods produced good results, although the symmetric formulation may be the better.

II. THE PHYSICAL PROBLEM AND ITS MATHEMATICAL FORMULATION

In considering steady state heat flow in two dimensions, we will determine numerically the temperature distribution in a thin slice of material of infinite extent and uniform thermal properties, that has heat sources except in a bounded region I (Figure 1). Inside this inner region I, the thermal conductivity and capacity may vary, and there may be heat sources or holes. Either the temperature or the heat flux is specified on the boundary of the holes. All points outside the inner region will be called E, the outer region.

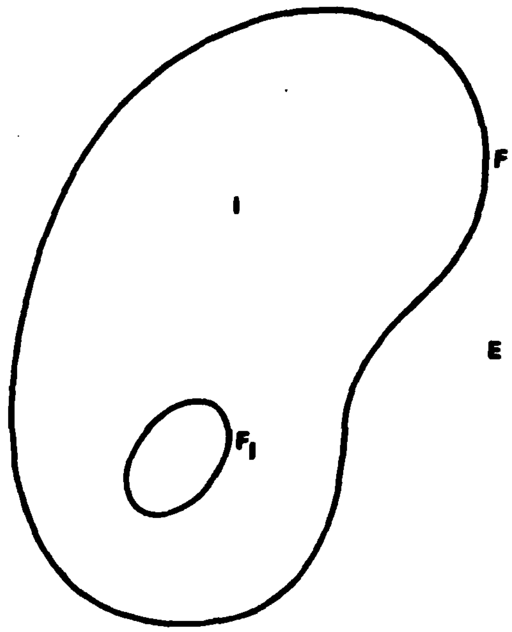

Figure 1 - Diagram of the Inner and Outer Regions

The physical problem is described analytically by partial differential equations that hold in the two regions and by the prescribed data on boundaries of the regions. The interface between the inner and outer regions is called F and the boundaries of any holes in the inner region are called F_I. The temperature distribution u is governed by the differential equations

$$\nabla^2 u = f, \text{ in I} \tag{1a}$$
$$\nabla^2 u = 0, \text{ in E} \tag{1b}$$

where f is the distribution of heat sources in I. On the boundaries of holes in I, the temperature and heat fluxes satisfy the boundary conditions

$$u = g, \text{ on } F_g \tag{2a}$$

$$\frac{\partial u}{\partial n} = h, \text{ on } F_h \tag{2b}$$

The boundary of the holes is composed of the disjoint parts F_g and F_h on which are prescribed the temperature and the heat flux, respectively. The derivative in the direction of the normal vector n pointing outward from the region is $\partial/\partial n$. With only a little extra complication, we could also consider boundary conditions of the form $Au + B \frac{\partial u}{\partial n} = C$, which can be used to represent convection at the boundary.[1*]

III. THE FINITE ELEMENT METHOD

The finite element method is used to solve for the temperature distribution in the inner region which has the more complex configuration of thermal properties. To apply the finite element method, the region is divided into elements that are small enough to accommodate changes in thermal properties and the distribution of heat sources and to adequately follow the shape of the boundaries. In practice a set of grid points, called nodes, is defined in the region; these nodes determine the sides of each element. Techniques of finite element modeling and details of the method are not covered in this report; Zienkiewicz[2] provides a comprehensive treatment.

The finite element method produces a matrix equation that relates the temperature distribution in the region with sources of heat in and on the boundary of the region and properties of the material in the region. This matrix equation is of the form

$$K\bar{u} = \bar{f} \tag{3}$$

where K is a matrix determined by the configuration of the finite elements and the thermal properties of the material, $\bar{u}$ is the vector of nodal temperatures,

*A complete listing of references is given on page 29.

and $\bar{f}$ is a vector of nodal heat sources determined by sources inside the region and heat fluxes on the boundary.

The matrix equation is modified by boundary conditions. For a node at which the temperature is known, the component of $\bar{u}$ for the node is constrained to equal the known value, and for the node at which $\partial u/\partial n$ is known, terms are added to the source vector $\bar{f}$. On the interface between the inner and outer regions, the boundary values serve to couple the solutions for the two regions. The way that this coupling is done will be described in Section V.

IV. THE BOUNDARY INTEGRAL EQUATION METHOD

The boundary integral equation method is used in the coupled problem to account for the effects of the infinite outer region and to compute the temperature at selected points in the region. The method reduces the problem of solving Equation (1b) for all points in the region to the problem of solving an integral equation on the boundary of the region; this is especially helpful in treating the infinite outer region. Use of the method will first be described for a bounded region and will then be extended to an unbounded region outside a curve. This description is offered to assist in understanding the method and will not be given in detail. Improvements and refinements to the method have been described by Zienkiewicz,[2] Butterfield,[3] and Brebbia.[4]

THE INTERIOR BOUNDARY INTEGRAL EQUATION METHOD

In the context of two-dimensional steady state heat flow, the problem is to determine the temperature distribution in the region D (see Figure 2) inside a curve F when the temperature is known on part of the boundary F_g and the heat flow is known on the remainder of the boundary F_h. That is, the differential equation, Equation (1b), with the boundary conditions, Equations (2), is to be solved by reducing the problem to solving an integral equation on F.

To obtain the boundary integral equation we begin with a form of Green's theorem for the region D:

$$\int_D (w\nabla^2 v - v\nabla^2 w)dA = \int_F \left(v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n}\right)ds$$

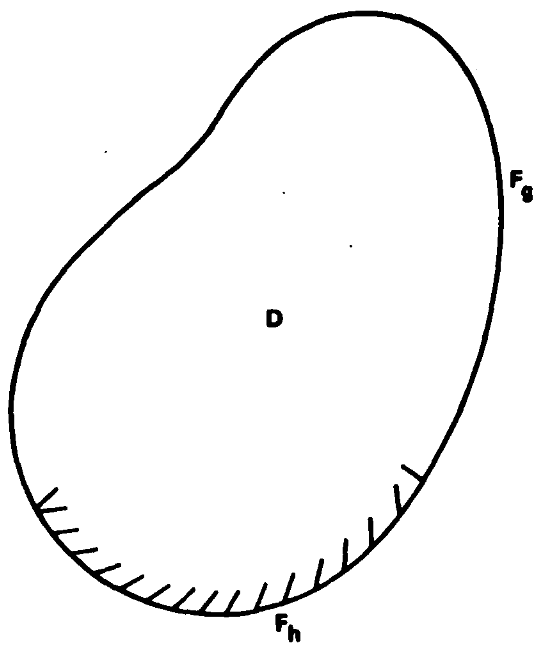

Figure 2 - The Bounded Region

in which ∇^2 is the Laplace operator $(\partial/\partial x)^2 + (\partial/\partial y)^2$ and $\partial/\partial n$ is the derivative in the direction of the normal pointing outward from D. We will derive the needed integral equations by substituting two particular functions for the functions v and w in Green's theorem. Equation (1b) and the boundary conditions (Equations (2)) form a well posed problem having a continuous solution u which we substitute for w. For v, we substitute the function $G(p,g) = -\frac{1}{2\pi}\log|p-q|$, where p and q are points in D; p is the point at which the value of the solution u is obtained and q is the variable of integration. This form of G(p,q) is the fundamental solution for Equation (1) in two-dimensions. With these substitutions Green's theorem becomes (see Appendix A)

$$u(p) = \int_F \left(u(q)\,\frac{\partial G(p,q)}{\partial n} - G(p,q)\,\frac{\partial u(q)}{\partial n} \right) ds(q) \qquad (4)$$

For any point p in D, the functions G(p,q) and $\partial G(p,q)/\partial n$ are explicity known for q on the curve F. If both boundary conditions u and $\partial u/\partial n$ were known at all points on the boundary, u(p) could be computed by a straightforward integration. However, for a well posed problem exactly one of the two boundary

conditions must be specified on any interval of the boundary so in the context of Equation (4) the other boundary values are missing on this interval. The next step (which is the first step in the numerical implementation) is to obtain the missing boundary data.

To obtain this missing data, we move the point p to the boundary. This must be done carefully (see Appendix B), because for p on F, the integrand becomes singular, although integrable. For p on F Equation (4) becomes

$$\frac{1}{2}\,u(p) = \int_F \left(u(q)\,\frac{\partial G(p,q)}{\partial n} - G(p,q)\,\frac{\partial u(q)}{\partial n} \right) ds(q) \qquad (5)$$

This is the boundary integral equation. If u is known on the entire boundary, that is, if we have a Dirichlet problem, $\partial u/\partial n$ is unknown and the boundary integral equation can be written

$$\int_F G(p,q)\,\frac{\partial u}{\partial n}\,ds(q) = \int_F u(q)\,\frac{\partial G(p,q)}{\partial n}\,ds(q) - \frac{1}{2}\,u(p)$$

The right side and G(p,q) are explicitly known. This integral equation is solved for $\partial u/\partial n$, which is then substituted in Equation (4) to yield temperatures at any point p. In a similar manner if $\partial u/\partial n$ is known on the boundary or if u is known on part of the boundary and $\partial u/\partial n$ is known on the remainder, the boundary integral equation can be used to obtain the missing boundary values.

THE EXTERIOR BOUNDARY INTEGRAL METHOD

To solve the heat conduction problem in an infinite region, we will couple the finite element solution in a bounded part of the region to the boundary integral solution outside of the bounded part. Therefore, a boundary integral solution for the region outside of a curve F is needed.

To compute the solution u to Equation (1b) at a point p in the region outside a closed curve, we begin by enclosing both the point p and the curve F in a circle S having radius R (Figure 3). Connecting the circle and the curve F with a cut C produces a region to which the preceding work applies. The

integral equation for points inside the resulting contour and for points on the curve F takes the form

$$au(p) = \left[\int_F + \int_{C_+} + \int_{C_-} + \int_S\right] \left(G(p,q)\,\frac{\partial u(q)}{\partial n} - u(q)\,\frac{\partial G(p,q)}{\partial n}\right) ds(q)$$

where a = 1 if p is inside the contour and a = 1/2 if p is on the curve F and F has a smoothly varying tangent at p. (Also see Appendix A.)

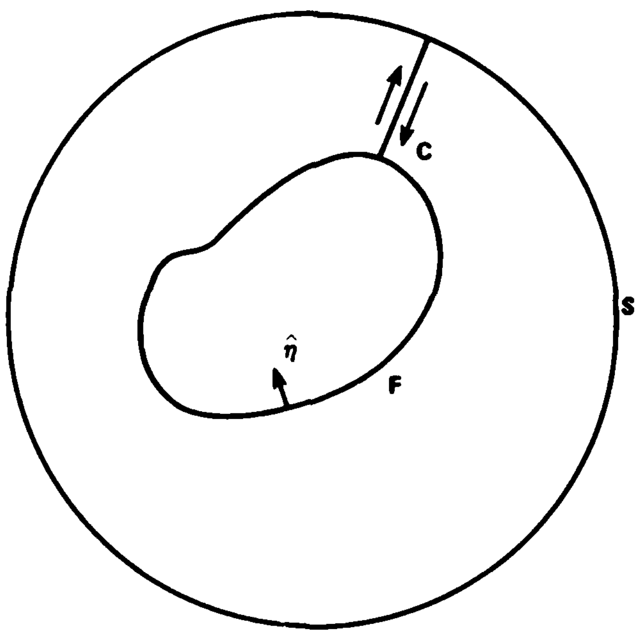

Figure 3 - Contours Used to Solve the Exterior Problem

In physical problems, the values of temperature, sound intensity, magnetic field strength, etc. go to zero at points far from any sources. In mathematical terms, this means we can restrict our attention to functions that satisfy the condition $u(q) = O(1/q)$ as $q \to \infty$. Then the preceding integral converges to an integral over the curve F as the radius R of the circle S becomes infinite. The convergence follows from two facts: (1) The integral over C_+ equals the integral over C_- except that the normal vector $\hat{n}$ points in the

opposite direction. Thus one integrand is the negative of the other and the integrals cancel. (2) Since on the circle S, $u(q) = O(1/R)$, $\partial u(q)/\partial n = \partial u(q)/\partial r = O(1/R^2)$ and $\lim_{R\to\infty} \frac{\log R}{R} = 0$,

$$\lim_{R\to\infty} \int_S \left(\frac{\partial u(q)}{\partial n} \log|p-q| - u(q) \frac{\partial}{\partial n} \log|p-q| \right) ds(q) = 0$$

Thus Equations (4) and (5) hold for points p outside the curve F if the normal vector $\hat{n}$ points inward, that is, away from the exterior region.

THE COMPUTATIONAL SOLUTION

Equations (4) and (5) define a solution to the boundary integral equation problem, but except for a few specially shaped boundaries explicit solutions are not available. Therefore a means of obtaining approximate numerical solutions must be used. To avoid unnecessary complications an approach in which the temperature and heat flux on the boundary are approximated by linear functions is used here. A more flexible and accurate method, which uses a representation of the boundary segments similar to that used in isoparametric finite elements, is described by Fairweather et al.[1]

Equations (4) and (5) are discretized by dividing the curve F into k segments $F_1, \ldots, F_k$ in such a way that on any interval F_i u and $\partial u/\partial n$ can be approximated by linear functions and that on any one interval either u or $\partial u/\partial n$ is given as boundary data. Let u_i and u_i' be the approximations to the values of u and $\partial u/\partial n$ on the first end point s_i of the segment F_i. The end points are ordered by traversing the boundary in the direction that keeps the exterior of the boundary integral region on the left. Let $N_i(s)$ be the linear function on the segments F_{i-1} and F_i that equals unity at s_i, equals zero at s_{i+1} (at s_2 and s_k, if i=1), and equals zero on all other segments of the boundary. Then the linear approximations are

$$u(s) \approx \sum_{i=1}^{k} u_i N_i(s) \tag{6a}$$

$$\frac{\partial u(s)}{\partial n} \approx \sum_{i=1}^{k} u_i' N_i(s) \tag{6b}$$

Substituting these approximations into Equations (4) and (5) produces the discretized equation

$$au_j + \sum_{i=1}^{k} u_i \int_F N_i(s) \frac{\partial G(p_j,q)}{\partial n} \, ds(q) = \sum_{i=1}^{k} u_i' \int_F N_i(s) \, G(p_j,q) \, ds(q) \tag{7}$$

where $a = 1/2$ if p_j is on the curve F and F has a continuously turning tangent at p_j (see Appendix A). For p_j in the interior of the boundary integral region, $a = 1$.

Since $G(p,q)$ and $N_i(s)$ are known explicitly, numerical quadrature gives numerical values for the matrix components

$$\tilde{A}_{ij} = \int_{F_i+F_{i-1}} N_i(s) \frac{\partial G(p_j,q)}{\partial n}$$

$$B_{ij} = \int_{F_i+F_{i-1}} N_i(s) \, G(p,q) \, ds(q)$$

For i=j, the integrand is singular, but its value can be found by an analytic approximation. The matrix A is defined by $A_{ij} = A_{ij}$ if $i = j$, and $A_{ii} = 1/2 + \tilde{A}_{ii}$. Combining these matrix components with the integral Equation (7) produces the matrix equation

$$A\bar{u} = B\bar{u}' \tag{8}$$

where $\bar{u} = (u_1, \ldots, u_k)$ and $\bar{u}' = (u_1', \ldots, u_k')$ are the vectors of temperatures and normal derivatives at the end points of the segments. This is the matrix form of the boundary integral equation.

Recall that at each point of the boundary either u or $\partial u/\partial n$ is known and the other is to be determined, so that exactly half of the components $u_1, \ldots, u_k, u_1', \ldots, u_k'$ are known. The remainder are determined by exchanging corresponding columns between the left and right sides of the matrix equation as necessary to move the unknown components to the left side and the known components to the right. This procedure yields a matrix equation of the form

$$Mx = b$$

that is solved for the unknown boundary values. When these boundary values have been computed, all the boundary values are known at all the segment end points on the boundary F.

Now that both u and $\partial u/\partial n$ are known on the boundary, we can use Equation (7) to determine the temperature at any point p_j that is not on the boundary. This time we use the matrix components $\tilde{A}_{ij}$ and B_{ij} to get the equation

$$u(p_j) = - \sum_{i=1}^{k} u_i \tilde{A}_{ij} + \sum_{i=1}^{k} u_i' B_{ij}$$

The components $\tilde{A}_{ij}$ and B_{ij} have the same form for p_j not on the boundary as the corresponding components have for p_j on the boundary, but in general, they are not equal to any of these components and, in fact, must be recomputed for each point that is not on the boundary.

V. COUPLING THE BOUNDARY INTEGRAL EQUATION AND FINITE ELEMENT SOLUTIONS

We wish to solve a heat flow problem using a finite element representation in a bounded region and a boundary integral equation representation of the surrounding infinite region. The individual methods have been described; it remains to couple the solutions. Two formulations of the coupling will be described. The first formulation, unsymmetric coupling, has the advantage that the boundary integral matrices A and B given in Equation (8) can be used directly without the need of any matrix inversion. It has the disadvantage that the coupling introduces unsymmetric matrices which cause the entire linear system to require an unsymmetric solution method. This lack of symmetry increases the effort needed to solve the system of equations. The advantage of the second formulation, symmetric coupling, is that the coupled linear system remains symmetric, so faster solution routines can be used to solve the system of equations. The disadvantage of this method is that the boundary integral matrix B must be inverted. If this matrix is small enough compared to the finite element matrix, the effort saved in avoiding an unsymmetric solution may make the effort of inverting the matrix worthwhile. Further details and some variants of these coupling formulations are described by Kelley et al.[5]

UNSYMMETRIC COUPLING

The unsymmetric coupling formulation can be thought of as a finite element solution for the inner region with special boundary conditions applied at the interface. The boundary integral equation method produces a relation between the temperature and heat flux on the interface that provides the required boundary conditions.

The finite element matrix Equation (3) is written expressing the source term as the sum of two terms. The first term represents the heat flux through the interface into the inner region due to the temperature distribution in the outer region. The equation is

$$K\bar{u} = \bar{f}_c + \bar{f}$$

The i-th component of $\bar{f}_c$ is

$$(f_c)_i = \int_F N_i \frac{\partial u}{\partial n} ds$$

where the N_i are the linear functions (shape functions) defined on page 9 (see Zienkiewicz[2]). The derivative $\partial u/\partial n$ is also expanded in terms of the N_i (Equation (6b)) and the preceding equation becomes

$$(f_c)_i = \sum_{i=1}^{k} u_i' \int_F N_i N_j \, ds$$

We define the matrix components

$$C_{ij} = \int_F N_i N_j \, ds \tag{9}$$

and write the finite element matrix equation in the form

$$K\bar{u} = C\bar{u}' + \bar{f}$$

Now we need to relate the nodal temperatures $\bar{u}$ with the nodal fluxes $\bar{u}'$ in this equation.

The boundary integral matrix Equation (8) provides the needed relation between the nodal temperatures and fluxes on the interface:

$$A\bar{u} = B\bar{u}'$$

The preceding two matrix equations are to be merged to form the coupled system. Let $u_1, \ldots, u_k$ and $u_1', \ldots, u_k'$ be the nodal temperatures and fluxes on the interface, and let $u_{k+1}, \ldots, u_m$ be the temperatures at the remainder of the finite element nodes. Merging the two matrix equations produces the following coupled finite element-boundary integral system of equations:

$$\left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{K}} & C \\ \hdashline & & 0 \\ \hdashline A & \multicolumn{1}{:c:}{0} & -B \end{array}\right] \left[\begin{array}{c} u_1 \\ \vdots \\ u_k \\ \hdashline u_{k+1} \\ \vdots \\ u_m \\ \hdashline u_1' \\ \vdots \\ u_k' \end{array}\right] = \left[\begin{array}{c} f_1 \\ \vdots \\ f_m \\ \hdashline 0 \end{array}\right] \tag{10}$$

This is the linear system of equations that is assembled and solved by a computer program to yield temperatures at nodes in the inner region and temperatures and fluxes on the interface.

SYMMETRIC COUPLING

The symmetric coupling formulation can be thought of as a more unified approach. It is derived by minimizing an energy integral for both the inner and outer regions. Both the finite element and boundary integral equation methods contribute terms to the energy integral.

The integral for the energy in the inner and outer regions and for energy arising from sources on the boundaries of holes in the inner region is

$$\begin{aligned} J(u) &= \frac{1}{2}\int_E (\nabla^2 u)\,dA + \frac{1}{2}\int_I (\nabla u)^2 dA - \int_{F_h} uh\,ds \\ &= \frac{1}{2}\int_F u\frac{\partial u}{\partial s}\,ds + \frac{1}{2}\int_I (\nabla u)^2 dA - \int_{F_h} uh\,ds \end{aligned}$$

The second line in this equation follows from Green's theorem and from $\nabla^2 u = 0$ in the region E.

To obtain a matrix form of the energy integral we will make several substitutions of approximate discretized expressions. The second term in the energy integral is approximated by the expression

$$\bar{u}^T K \bar{u}$$

where K is the finite element stiffness matrix for the inner region.[2] In the first and third terms, we substitute the expansions (Equations (6a), (6b)) of u and $\partial u/\partial n$ in terms of shape functions and nodal temperatures and fluxes, now written in vector notation with $N = (N_1, \ldots, N_k)$

$$\begin{aligned} u &\approx N\bar{u} \\ \frac{\partial u}{\partial n} &\approx N\bar{u}' \end{aligned}$$

The vector of fluxes $\bar{u}'$ can be expressed in terms of the temperatures $\bar{u}$, using the boundary integral matrix Equation (8) solved for $\bar{u}'$:

$$\bar{u}' = B^{-1}A\bar{u} \tag{11}$$

so that

$$\frac{\partial u}{\partial n} \approx NB^{-1}A\bar{u}$$

Also, we define the discretized source vector

$$H = \left(\int_{F_h} N_i h ds \right)$$

and use the matrix C defined by Equation (9). With these approximations, the energy integral can be written in matrix form

$$J(\bar{u}) = \frac{1}{2} u^T CB^{-1} A\bar{u} + \frac{1}{2} \bar{u}^T K\bar{u} - H\bar{u}$$

Now minimizing the energy integral $J(\bar{u})$ by setting its variations with respect to u_i equal to zero yields the matrix equation for the temperature vector $\bar{u}$

$$\frac{1}{2} (C^T B^{-1} A + (B^{-1} A)^T C)\bar{u} + K\bar{u} = H \tag{12}$$

This is the symmetric system of linear equations that is assembled and solved by the computer program to yield temperatures at nodes in the inner region and on the interface. If the fluxes are needed on the interface, they are easily obtained by saving the previously computed matrix $B^{-1}A$ and using Equation (11).

The problem of computing the coupled finite element-boundary integral solution is reduced to solving either of the systems of Equations (10) or (12). Next, we develop the computer methods used to solve these systems.

VI. THE COMPUTER IMPLEMENTATION

The solution method described in the preceding section is implemented using two existing computer programs. A boundary integral equation program which employs linear approximations is used to form the matrices A and B.[6] This program is easily modified to save the matrices in a form that can be passed to the finite element program. The computation of the matrix C (Equation (9)) is added to the boundary integral program. The NASTRAN computer program solves rather general finite element problems and therefore can represent the inner region which requires detailed representation. In addition, NASTRAN will easily accept and manipulate the boundary integral matrices and incorporate the results into the solution of the finite element system of linear equations. At the completion of the finite element solution, the

vectors of nodal temperatures and heat fluxes are passed back to the boundary integral program for use in computing temperatures required in the outer region using Equation (4).

NASTRAN contains sequences of computations which are executed by the program to carry out finite element solutions. These sequences, called rigid formats, are easily altered to allow the introduction of variations in the finite element solution procedures. A sequence of these rigid format ALTERs is used for each of the two formulations that we are considering. The data used by NASTRAN consist of the finite element representation of the inner region, the matrices A, B, and C, and the list of rigid format ALTERs which give the changes used to incorporate the boundary integral matrices into the finite element solution. For details of the use and meaning of the rigid format ALTERs see the NASTRAN User's Manual.[7]

We now give a listing and description of the rigid format ALTERs used for the unsymmetric coupling formulation.

```
ALTER 37
 INPUTT2  /A,B,C,,/0/11  $
 PARTN  HKGG,PVL,/KSML,,  $
 MERGE, ,,A,,PVS,/AO/1  $
 MERGE, ,C,,,,PVS/CO/1  $
 MERGE  KSML,AO,CO,B,PVL,PVL/KCPL/1  $
 EQUIV  KCPL,HKGG  $
ALTER  79,80
 DECOMP HKLL/L,U  $
ALTER  93,99
 FBS  L,U,HPL/HULV  $
ALTER 108
 PARTN  HUGV,,PVU/,PHI,,/0  $
 PARTN  HUGV,,PVL/,DPHI,,/0  $
 OUTPUT2  PHI,DPHI,,,//-1/11  $
ENDALTER
```

The INPUTT2 instruction reads the matrices A, B, and C. To allow room for the matrix in Equation (10), as many extra degrees of freedom are generated as there are nodes on the interface, so the finite element matrix for the inner

region is smaller than the matrix in NASTRAN. The first PARTN instruction partitions the larger matrix HKGG and extracts the matrix for the inner region which is called KSML. The three MERGE instructions assemble the matrix in Equation (10) and call it KCPL. The EQUIV instruction equates the newly generated, coupled matrix with HKGG, the matrix whose equation is to be solved as the finite element computations continue. The DECOMP and FBS instructions ensure that unsymmetric decomposition routines are used to solve the system of linear equations. After the finite element solution is complete, the two PARTN and the OUTPUT2 instructions extract the temperatures and heat fluxes on the interface and save them for use by the boundary integral equation program.

For the symmetric coupling formulation, the following rigid format ALTERs are used:

```
ALTER 37
 INPUTT2  /A,B,C,,/0/11  $
 SOLVE  B,A/E  $
 MPYAD  C,E,/CTE/C,N,1  $
 TRNSP  CTE/ETC  $
 ADD  CTE,ETC/KPR/C,N,(.5,0.)/C,N,(.5,0.)  $
 MERGE,  ,,,KPR,PVU,/KBI///C,N,6  $
 ADD  HKGG,KBI/KCPL  $
 EQUIV  KCPL,HKGG  $
ALTER 108
 PARTN  HUGV,,PVU/,PHI,,/0  $
 MPYAD  E,PHI,/DPHI/C,N,0  $
 OUTPUT2  PHI,DPHI,,,//-1//11  $
ENDALTER
```

The INPUTT2 instruction reads the matrices A, B, and C. The SOLVE, MPYAD, and TRNSP instructions form the matrix $(C^TB^{-1}A + (B^{-1}A)^TC)/2$ for Equation (12); this matrix is called KPR. The MERGE instruction increases the dimension of KPR so it can be added to the finite element matrix HKGG by the ADD instruction which forms the coupled boundary integral-finite element matrix KCPL. The EQUIV instruction sets the finite element matrix HKGG equal to the coupled

matrix KCPL and the finite element solution continues. When the solution is complete, the PARTN instruction extracts the nodal temperatures on the interface, and the MPYAD instruction produces the nodal fluxes using the matrix E computed earlier by the SOLVE instruction according to Equation (11). The OUTPUT2 instruction saves the nodal temperatures and fluxes for use by the boundary integral program.

If temperatures at points in the outer region are desired, the boundary integral program uses the nodal temperatures and fluxes on the interface that are saved by NASTRAN to compute the temperatures according to Equation (4).

VII. THE SAMPLE PROBLEM

A steady state, two-dimensional heat conduction problem was solved to illustrate the coupled finite element boundary integral method. The inner and outer regions are shown in Figure 4. The irregular shape was chosen to avoid having geometrical symmetry produce better results than would otherwise have been obtained.

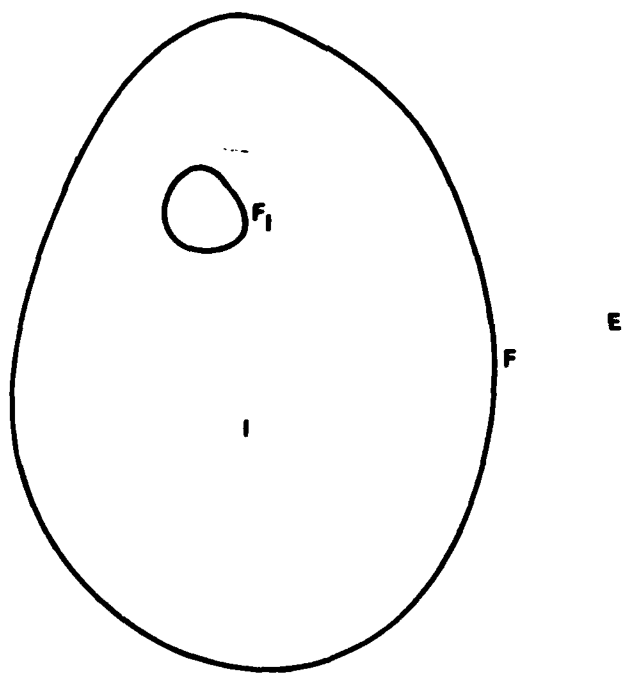

Figure 4 - Regions and Boundaries for the Sample Problem

The sample problem is an exterior Dirichlet problem. The temperature is prescribed on the interior boundary F, consistent with the function $u(r,\Theta) = \cos \Theta/r$, where r is the distance from the point 0 inside the boundary F, and Θ is the polar angle. Since u is a known solution of Equation (1), temperature and flux values can be calculated at any point to compare with the computed solution.

Two representations of the inner region and the interface in terms of finite elements and boundary segments were used; these representations are shown in Figures 5a and 5b. The grid in Figure 5b has twice as many boundary segments as the grid in Figure 5a, but except for adjustments needed along the interface to accommodate the extra points, the finite element grids are the same. It is apparent from Figure 5 that in the grid with fewer boundary segments the length of the segments is more compatible with the finite element mesh in that an element refinement along the interface that accommodates only the boundary segments is not required.

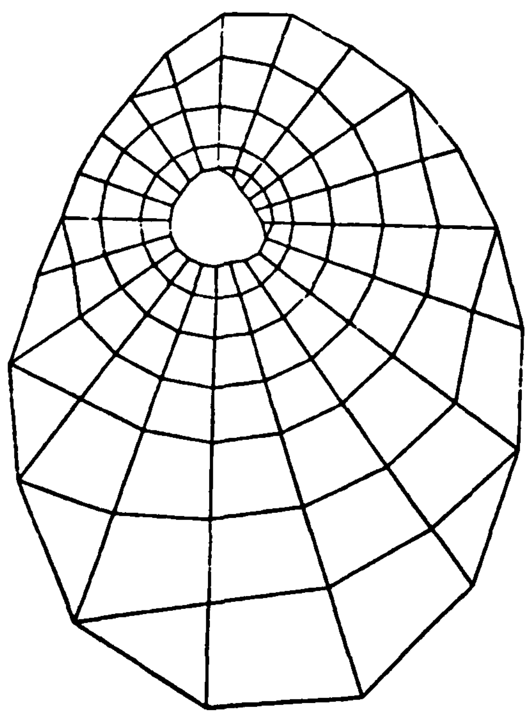

Figure 5a

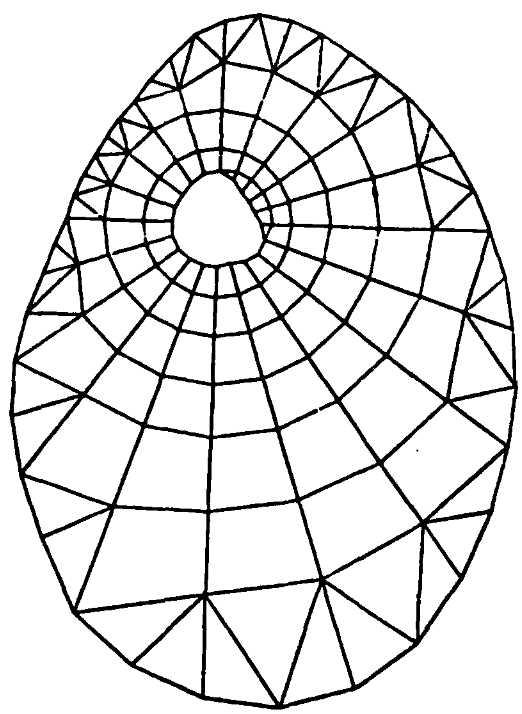

Figure 5b

Figure 5 - Boundary Integral and Finite Element Grid for the Sample Problem

Results from the sample problem are tabulated in Appendix B. Four trends can be seen in these results:

- The division of the interface into segments shorter than those consistent with the finite element mesh did not produce more accurate results except in the outer region. Even there the errors are not significantly smaller.
- There is no significant difference between the errors from the symmetric and unsymmetric formulations, except in the outer region where the errors from the symmetric formulation average a little more than one-half of those from the unsymmetric formulation.
- The errors in both the temperatures and fluxes on the interface become larger on the part of the interface where there are fewer finite elements between the inner boundary and the interface.

  The largest errors encountered are in the temperatures and fluxes on the interface. Even though the values on the interface are subsequently used to compute temperatures in the outer region, the errors in the outer region are quite small.

The reasons for these trends have not been studied yet, but some conjectures can be made. It is likely that, with consistent boundary segment lengths and finite element meshes, the two methods produce the same order of error, so that reducing the error produced by one part of the procedure does not improve the overall accuracy of the coupled method. The approximations made in both the boundary integral equation and finite element methods are the same for the symmetric and unsymmetric formulations, so it would be expected that the results would be nearly equal. Temperatures in the outer region are computed by integrating temperatures and fluxes on the interface. The smoothing characteristic of integration tends to average positive and negative errors on the interface, producing results in the outer region with smaller errors.

The results of this admittedly small sample indicate that the method produces generally good values and very good values in the outer region. The use of boundary segments having lengths compatible with the finite element mesh and the use of the symmetric formulation are recommended when the coupled method is used.

APPENDIX A - THE BOUNDARY INTEGRAL EQUATION

For u the solution of the differential equation, Equation (1b), that satisfies the boundary conditions, Equation (2), we derive Equation (4) and Equation (5) for p in the region D and for p on the boundary of D, respectively. These derivations are well known but not often explicitly written. Since the location of the interface between the inner and outer regions can be chosen so that u is harmonic in a domain that extends a finite distance beyond D, u can be assumed to be continuous and $\partial u/\partial n$ to be bounded in and on the boundary of D.[8]

For p in D, $\frac{1}{2\pi} \log |p-q|$ as a function of q satisfies the Laplace equation except at the point p. A small disk, with center p and radius r bounded by the circle K_r is removed from D, leaving the modified region D_r (Figure 6). On D_r, Green's theorem holds

$$\begin{aligned} &\frac{1}{2\pi}\left[\int_F + \int_{K_r}\right]\left(\frac{\partial u}{\partial n} \log|p-q| - u \frac{\partial}{\partial n} \log|p-q|\right) ds(q) \\ &= \frac{1}{2\pi}\int_{D_r} (\nabla^2 u \log|p-q| - u\nabla^2 \log|p-q|) dA(q) \end{aligned} \tag{13}$$

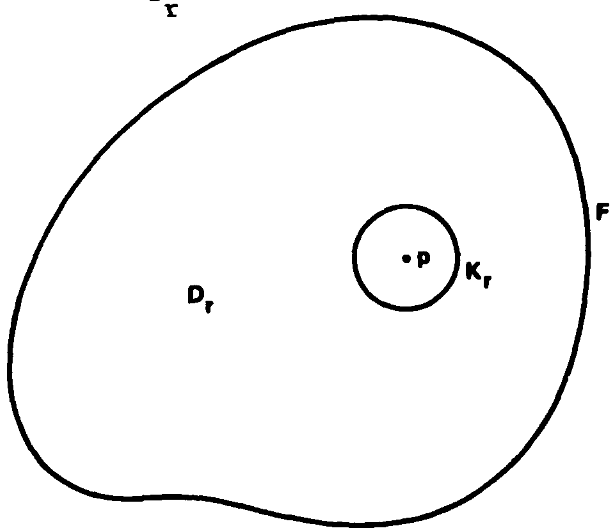

Figure 6 - The Modified Region D_r for an Interior Point

In the modified region D_r, both u and log $|p-q|$ are harmonic, so the right side in the preceding equation is zero. In a local coordinate system with origin at p, the integral over K_r becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \left(-\frac{\partial u}{\partial r} \log r + \frac{u}{r}\right) r \, d\theta$$

since on K_r, $|p-q| = r$ and $\partial/\partial n = -\partial/\partial r$.

In the limit as $r \to 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial n}\, r \log r \, d\theta \to 0$$

$$\frac{1}{2\pi}\int_0^{2\pi} u \, d\theta \to u(p)$$

and Equation (13) becomes Equation (4):

$$u(p) = \int_F \left(u \frac{\partial}{\partial n} \log|p-q| - \frac{\partial u}{\partial n} \log|p-q|\right) ds(q)$$

If p is on the boundary of D and the tangents at p form the angle a (Figure 7), a similar derivation provides a slightly different equation. This time we remove a small sector with vertex p and radius r and bounded by the arc A_r, again leaving a modified region D_r. In the same manner as before, taking a limit and applying Green's theorem yields

$$\lim_{r\to 0} \frac{1}{2\pi}\int_0^{a} \left(-\frac{\partial u}{\partial r}\log r + \frac{u}{r}\right) r \, d\theta = \frac{a}{2\pi} u(p)$$

and

$$\frac{a}{2\pi} u(p) = \int_F \left(\frac{\partial u}{\partial n} \log|p-q| + u \frac{\partial}{\partial n} \log|p-q|\right) ds(q)$$

This result is more general than Equation (5), which is obtained for the case of a continuously turning tangent at p, that is, $a = \pi$.

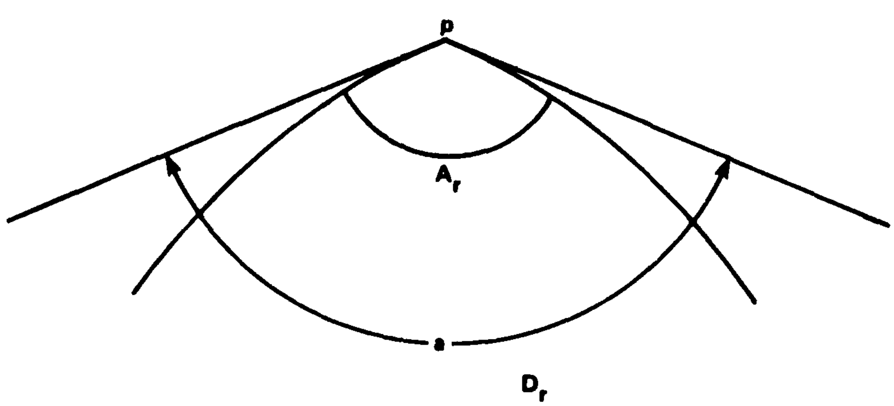

Figure 7 - The Modified Region D_r for a Boundary Point

APPENDIX B - TABULATED RESULTS FROM THE SAMPLE PROBLEM

Tables 1 through 6 list temperatures, fluxes, and errors computed for the sample problem. These values were computed using the symmetric and unsymmetric formulations, and for each formulation using meshes with 20 and 40 grid points on the interface between the inner and outer regions (see Figure 5). Table 1 gives temperatures for a few points lying in the outer region (grid points numbered 1 to 6) and in the inner region (grid points numbered 7 to 20). Table 3 gives temperatures for points lying on the interface (grid points numbered 21 to 40). The normal derivatives or heat fluxes on the interface are given in Table 5. Tables 2, 4, and 6, which correspond to Tables 1, 3, and 5 contain relative errors in the computed temperatures and fluxes as percents of the maximum exact value for each group. Figure 8 shows the location of the grid points for which results are tabulated.

The following acronyms are used to designate the columns in the tables:

GP - grid point number
X,Y - grid point coordinates
U - exact temperatures
DU - exact fluxes
US - temperature from symmetric formulation
UU - temperature from unsymmetric formulation
DUS - flux from symmetric formulation
DUU - flux from unsymmetric formulation
ES - temperature error from symmetric formulation
EU - temperature error from unsymmetric formulation
DES - flux error from symmetric formulation
DEU - flux error from unsymmetric formulation
20,40 - refer to grids with 20 or 40 boundary segments

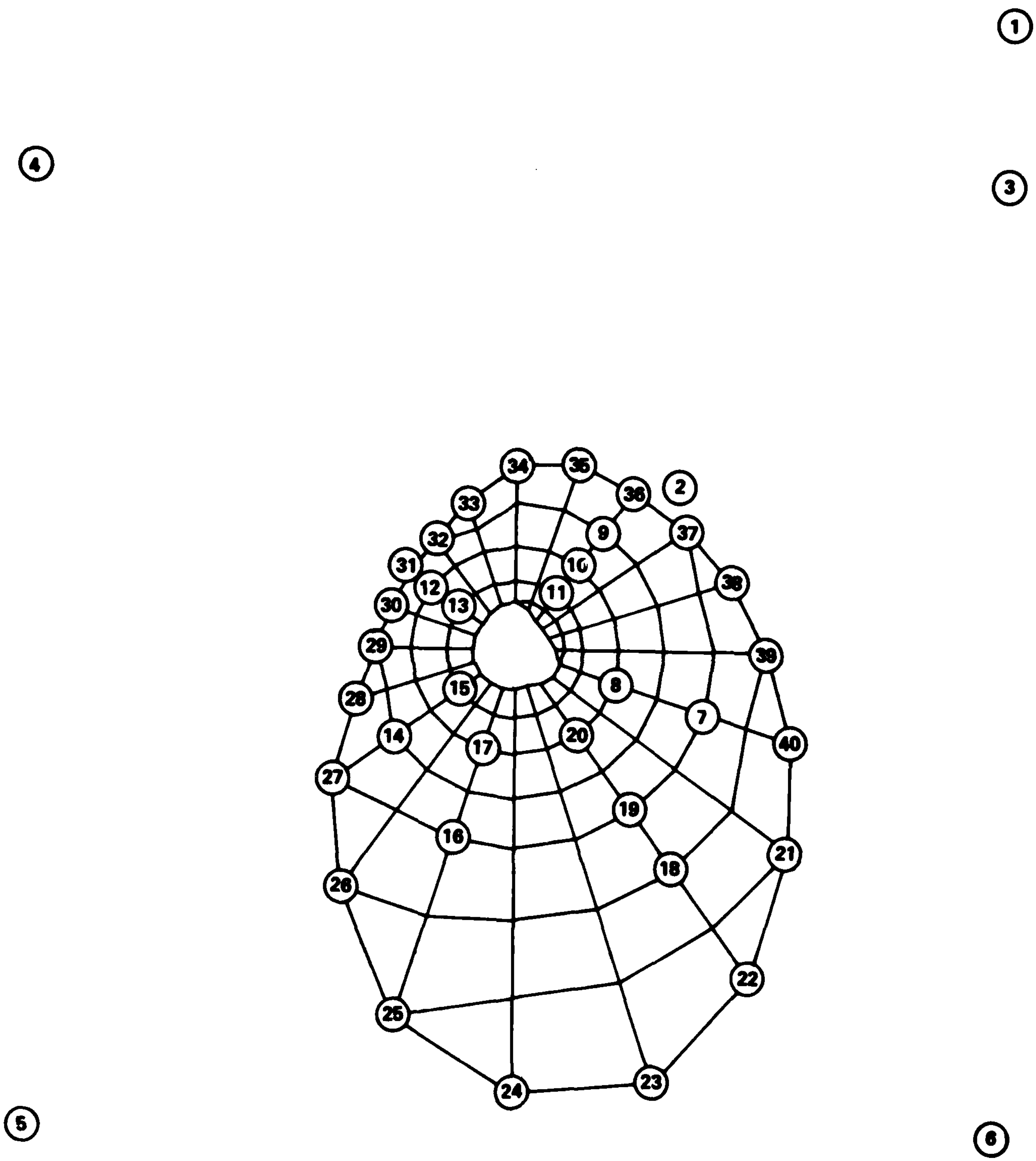

Figure 8 - Grid Point Locations for Tabulated Results from the Sample Problem

TABLE 1 - TEMPERATURES IN THE INNER AND OUTER REGIONS

GP	X	Y	U	US40	US20	UU40	UU20
1	6.00	8.00	0.0600	0.0700	0.0735	0.0778	0.0805
2	2.00	2.00	0.2500	0.2638	0.2651	0.2673	0.2679
3	6.00	6.00	0.0833	0.0939	0.0971	0.1011	0.1035
4	-6.00	6.00	-0.0833	-0.0805	-0.0771	-0.0730	-0.0699
5	-6.00	-6.00	-0.0833	-0.0809	-0.0776	-0.0737	-0.0708
6	6.00	-6.00	0.0833	0.0945	0.0977	0.1014	0.1036
7	2.26	-0.74	0.3996	0.4192	0.4177	0.4226	0.4201
8	1.19	-0.39	0.7608	0.7780	0.7771	0.7799	0.7786
9	1.05	1.45	0.3284	0.3459	0.3440	0.3482	0.3460
10	0.73	1.01	0.4702	0.4806	0.4794	0.4823	0.4808
11	0.48	0.66	0.7168	0.7120	0.7114	0.7130	0.7122
12	-1.01	0.73	-0.6472	-0.6715	-0.6662	-0.6699	-0.6633
13	-0.66	0.48	-0.9866	-0.9965	-0.9942	-0.9955	-0.9927
14	-1.45	-1.05	-0.4520	-0.4669	-0.4723	-0.4628	-0.4666
15	-0.66	-0.48	-0.9866	-1.0053	-1.0049	-1.0038	-1.0028
16	-0.74	-2.26	-0.1298	-0.1319	-0.1322	-0.1282	-0.1284
17	-0.39	-1.19	-0.2472	-0.2565	-0.2570	-0.2541	-0.2542
18	1.89	-2.61	0.1825	0.1945	0.1953	0.1985	0.1985
19	1.40	-1.93	0.2470	0.2592	0.2594	0.2626	0.2622
20	0.73	-1.01	0.4702	0.4864	0.4861	0.4884	0.4879

TABLE 2 - TEMPERATURE ERRORS IN THE INNER AND OUTER REGIONS - PERCENT OF MAXIMUM TEMPERATURE

GP	X	Y	ES40	ES20	EU40	EU20
1	6.00	8.00	1.0	1.4	1.8	2.1
2	2.00	2.00	1.4	1.5	1.8	1.8
3	6.00	6.00	1.1	1.4	1.8	2.0
4	-6.00	6.00	0.3	0.6	1.0	1.4
5	-6.00	-6.00	0.2	0.6	1.0	1.3
6	6.00	-6.00	1.1	1.5	1.8	2.1
7	2.26	-0.74	2.0	1.8	2.3	2.1
8	1.19	-0.39	1.7	1.6	1.9	1.8
9	1.05	1.45	1.8	1.6	2.0	1.8
10	0.73	1.01	1.1	0.9	1.2	1.1
11	0.48	0.66	0.5	0.5	0.4	0.5
12	-1.01	0.73	2.5	1.9	2.3	1.6
13	-0.66	0.48	1.0	0.8	0.9	0.6
14	-1.45	-1.05	1.5	2.1	1.1	1.5
15	-0.66	-0.48	1.9	1.9	1.7	1.6
16	-0.74	-2.26	0.2	0.2	0.2	0.1
17	-0.39	-1.19	0.9	1.0	0.7	0.7
18	1.89	-2.61	1.2	1.3	1.6	1.6
19	1.40	-1.93	1.2	1.3	1.6	1.5
20	0.73	-1.01	1.6	1.6	1.8	1.8

TABLE 3 - TEMPERATURES ON THE INTERFACE

GP	X	Y	U	US40	US20	UU40	UU20
21	3.28	-2.38	0.1998	0.2158	0.2136	0.2207	0.2174
22	2.82	-3.88	0.1225	0.1358	0.1371	0.1407	0.1406
23	1.67	-5.15	0.0570	0.0670	0.0699	0.0717	0.0731
24	0.00	-5.30	0.0000	0.0055	0.0088	0.0103	0.0127
25	-1.43	-4.39	-0.0669	-0.0670	-0.0611	-0.0628	-0.0577
26	-2.07	-2.85	-0.1670	-0.1731	-0.1677	-0.1683	-0.1628
27	-2.18	-1.59	-0.2996	-0.3111	-0.3044	-0.3065	-0.2990
28	-1.90	-0.62	-0.4755	-0.5102	-0.5089	-0.5033	-0.4993
29	-1.66	0.00	-0.6024	-0.6355	-0.6351	-0.6299	-0.6266
30	-1.49	0.49	-0.6058	-0.6329	-0.6350	-0.6303	-0.6315
31	-1.25	0.91	-0.5219	-0.5420	-0.5433	-0.5402	-0.5393
32	-0.96	1.32	-0.3606	-0.3714	-0.3610	-0.3712	-0.3602
33	-0.59	1.81	-0.1626	-0.1672	-0.1607	-0.1669	-0.1598
34	0.00	2.25	0.0000	0.0021	0.0077	0.0046	0.0115
35	0.74	2.26	0.1298	0.1394	0.1416	0.1422	0.1443
36	1.40	1.93	0.2470	0.2620	0.2652	0.2648	0.2675
37	2.04	1.48	0.3210	0.3372	0.3410	0.3401	0.3434
38	2.57	0.83	0.3522	0.3727	0.3753	0.3762	0.3774
39	3.00	0.00	0.3333	0.3531	0.3526	0.3572	0.3554
40	3.29	-1.07	0.2749	0.2936	0.2945	0.2982	0.2973

TABLE 4 - TEMPERATURE ERRORS IN THE INTERFACE - PERCENT OF MAXIMUM TEMPERATURE

GP	X	Y	ES40	ES20	EU40	EU20
21	3.28	-2.38	2.6	2.3	3.4	2.9
22	2.82	-3.88	2.2	2.4	3.0	3.0
23	1.67	-5.15	1.7	2.1	2.4	2.7
24	0.00	-5.30	0.9	1.5	1.7	2.1
25	-1.43	-4.39	0.0	1.0	0.7	1.5
26	-2.07	-2.85	1.0	0.1	0.2	0.7
27	-2.18	-1.59	1.9	0.8	1.1	0.1
28	-1.90	-0.62	5.7	5.5	4.6	3.9
29	-1.66	0.00	5.5	5.4	4.5	4.0
30	-1.49	0.49	4.5	4.8	4.0	4.2
31	-1.25	0.91	3.3	3.5	3.0	2.9
32	-0.96	1.32	1.8	0.1	1.7	0.1
33	-0.59	1.81	0.8	0.3	0.7	0.5
34	0.00	2.25	0.3	1.3	0.8	1.9
35	0.74	2.26	1.6	1.9	2.0	2.4
36	1.40	1.93	2.5	3.0	2.9	3.4
37	2.04	1.48	2.7	3.3	3.2	3.7
38	2.57	0.83	3.4	3.8	4.0	4.2
39	3.00	0.00	3.3	3.2	3.9	3.6
40	3.29	-1.07	3.1	3.2	3.8	3.7

TABLE 5 - HEAT FLUXES ON THE INTERFACE

GP	X	Y	DU	DUS40	DUS20	DUU40	DUU20
21	3.28	-2.38	0.0276	0.0365	0.0208	0.0373	0.0209
22	2.82	-3.88	0.0065	0.0110	0.0070	0.0109	0.0062
23	1.67	-5.15	0.0055	0.0073	0.0054	0.0064	0.0036
24	0.00	-5.30	0.0089	0.0074	0.0074	0.0064	0.0071
25	-1.43	-4.39	0.0138	0.0093	0.0161	0.0069	0.0138
26	-2.07	-2.85	0.0040	-0.0032	0.0055	-0.0034	0.0056
27	-2.18	-1.59	-0.0263	-0.0268	-0.0013	-0.0276	-0.0044
28	-1.90	-0.62	-0.1339	-0.2070	-0.1671	-0.1976	-0.1609
29	-1.66	0.00	-0.3410	-0.4075	-0.3637	-0.3946	-0.3561
30	-1.49	0.49	-0.3996	-0.4440	-0.4380	-0.4440	-0.4485
31	-1.25	0.91	-0.3156	-0.3446	-0.3689	-0.3508	-0.3638
32	-0.96	1.32	-0.1077	-0.1294	-0.0804	-0.1445	-0.0880
33	-0.59	1.81	0.0594	0.0364	0.0630	0.0220	0.0568
34	0.00	2.25	0.0659	0.0529	0.0757	0.0480	0.0792
35	0.74	2.26	0.0590	0.0581	0.0597	0.0562	0.0569
36	1.40	1.93	0.1131	0.1206	0.1243	0.1286	0.1229
37	2.04	1.48	0.1440	0.1484	0.1504	0.1462	0.1493
38	2.57	0.83	0.1371	0.1543	0.1499	0.1543	0.1478
39	3.00	0.00	0.1031	0.1163	0.1021	0.1174	0.1016
40	3.29	-1.07	0.0597	0.0724	0.0652	0.0740	0.0635

TABLE 6 - HEAT FLUX ERRORS ON THE INTERFACE -
PERCENT OF MAXIMUM FLUX

GP	X	Y	DES40	DES20	DEU40	DEU20
21	3.28	-2.38	2.2	1.7	2.4	1.7
22	2.82	-3.88	1.1	0.1	1.1	0.1
23	1.67	-5.15	0.5	0.0	0.2	0.5
24	0.00	-5.30	0.4	0.4	0.6	0.5
25	-1.43	-4.39	1.1	0.6	1.7	0.0
26	-2.07	-2.85	1.8	0.4	1.9	0.4
27	-2.18	-1.59	0.1	6.3	0.3	5.5
28	-1.90	-0.62	18.3	8.3	15.9	6.8
29	-1.66	0.00	16.6	5.7	13.4	3.8
30	-1.49	0.49	11.1	9.6	11.1	12.2
31	-1.25	0.91	7.3	13.3	8.8	12.1
32	-0.96	1.32	5.4	6.8	9.2	4.9
33	-0.59	1.81	5.8	0.9	9.4	0.7
34	0.00	2.25	3.3	2.5	4.5	3.3
35	0.74	2.26	0.2	0.2	0.7	0.5
36	1.40	1.93	1.9	2.8	3.9	2.5
37	2.04	1.48	1.1	1.6	0.6	1.3
38	2.57	0.83	4.3	3.2	4.3	2.7
39	3.00	0.00	3.3	0.3	3.6	0.4
40	3.29	-1.07	3.2	1.4	3.6	1.0

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